EECS 16B

4.9 LECTURE 22

STABILITY

In scalar, we saw that $|\lambda|<1$ is stable, all else is unstable.

Let's consider vectors: $\vec{x}(t+1) = A\vec{x}(t) + B\,u(t) \qquad \vec{x} \in \mathbb{R}^n$

look at eigenvals of A: $\lambda_1, \lambda_2, \dots \lambda_n \longrightarrow |\lambda_i|<1 \ \forall i$

If A is diagonalizable, we can find $\vec{z} = T\vec{x} \longrightarrow \vec{z}(t+1) = \underbrace{TAT^{-1}}_{A_{new}}\vec{z}(t) + \underbrace{TB}_{B_{new}}u(t)$

check for all $i \le n \longleftarrow z_i(t+1) = \lambda_i z_i(t) + B_{new,i}\,u(t)$

Even when A isn't diagonalizable, it can be brought to upper-diagonal form.

$$TAT^{-1} = \begin{bmatrix} \lambda_1 & \star & - & \star \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \star \\ 0 & - & 0 & \lambda_n \end{bmatrix} \longrightarrow \begin{bmatrix} z_1(t+1) \\ \vdots \\ z_n(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & \ddots & \star \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{bmatrix} + B_{new}\,u(t)$$

bounded if $|\lambda_n|<1 \longleftarrow z_n(t+1) = \lambda_n z_n(t) + B_{new,n}\,u(t)$

bounded if $|\lambda_{n-1}|<1 \longleftarrow z_{n-1}(t+1) = \lambda_{n-1} z_{n-1}(t) + \underbrace{\star\, z_n(t) + B_{new,n-1}\,u(t)}_{\text{bounded; treat as input}}$

↳ apply to all λ_i

If any λ_i fails this condition, the entire system is unstable.

CONTINUOUS STABILITY

scalar: $\frac{d}{dt}x(t) = \lambda x(t) + bu(t) \longrightarrow x(t) = e^{\lambda t}x(0) + b\int_0^t e^{\lambda(t-s)}u(s)\,ds$

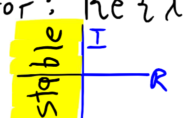
will this converge or diverge?

If $\lambda<0$, $e^{\lambda t} \to 0$ as $t \to \infty$. Also, bounded $u(\cdot)$ gives bounded $x(\cdot)$.

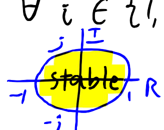
If $\lambda>0$, $e^{\lambda t} \to \infty$, so unstable for all nonzero $x(0)$.

If λ is complex: $\lambda = \alpha + j\omega \longrightarrow e^{\lambda t} = e^{\alpha t}\underbrace{e^{j\omega t}}_{\text{cosine}}$ need $\alpha = \mathrm{Re}\{\lambda\} < 0$.

vector: $\mathrm{Re}\{\lambda_i(A)\} < 0 \quad \forall i \in \{1, \dots, n\}$ in continuous time.



eigenvalues continuous



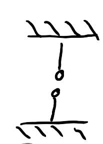
eigenvalues discrete

EX from lecture 6A: pendulum

$\frac{d}{dt}x_1(t) = x_2(t)$

$\frac{d}{dt}x_2(t) = -\frac{k}{m}x_2(t) - \frac{g}{l}\sin x_1(t)$

$f(\vec{x}) = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$ eq.: $x_1 = 0,\ x_2 = 0$; $x_1 = \pi,\ x_2 = 0$



$\nabla f(\vec{x}) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}$ eq.1: $A_{down} = \begin{bmatrix} 0 & 1 \\ -g/l & -k/m \end{bmatrix}$ eq.2: $A_{up} = \begin{bmatrix} 0 & 1 \\ g/l & -k/m \end{bmatrix}$

$\det(\lambda I - A_{down}) = \lambda^2 + \frac{k}{m}\lambda + \frac{g}{l} \longrightarrow \lambda_{1,2} = \frac{1}{2}\left(-\frac{k}{m} \mp \sqrt{\left(\frac{k}{m}\right)^2 - 4\frac{g}{l}}\right) \longrightarrow \mathrm{Re}\{\lambda_{1,2}\} < 0$

$\det(\lambda I - A_{up}) = \lambda^2 + \frac{k}{m}\lambda - \frac{g}{l} \longrightarrow \lambda_{1,2} = \frac{1}{2}\left(-\frac{k}{m} \mp \sqrt{\left(\frac{k}{m}\right)^2 + 4\frac{g}{l}}\right) \longrightarrow \mathrm{Re}\{\lambda_1\} < 0,\ \mathrm{Re}\{\lambda_2\} > 0$