EECS 16B ## 3.31 LECTURE 19

SVD REVIEW

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \dots + \sigma_r \vec{u}_r \vec{v}_r^T$$
$$= [\sigma_1 \vec{u}_1 \ \dots \ \sigma_r \vec{u}_r] \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_r^T \end{bmatrix}$$
$$= [\vec{u}_1 \dots \vec{u}_r] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_r^T \end{bmatrix}$$
$$\underbrace{A}_{m\times n} = \underbrace{U_1}_{m\times r} \ \underbrace{S}_{r\times r} \ \underbrace{V_1^T}_{r\times n}$$

$\vec{u}_1 \dots \vec{u}_r$ are evectors of AA^T
$\vec{v}_1 \dots \vec{v}_r$ are evectors of A^TA
respectively, $\vec{u}_{r+1} \dots \vec{u}_m$ has 0 evals
$\vec{v}_{r+1} \dots \vec{v}_m$ has 0 evals

$$U = [\underbrace{\vec{u}_1 \dots \vec{u}_r}_{U_1} \ \underbrace{\vec{u}_{r+1} \dots \vec{u}_m}_{U_2}] \quad m\times m$$
$$V = [\underbrace{\vec{v}_1 \dots \vec{v}_r}_{V_1} \ \underbrace{\vec{v}_{r+1} \dots \vec{v}_n}_{V_2}] \quad n\times n$$

Because the columns are orthonormal, $U^TU = I^m$ and $V^TV = I^n$

$$A = U_1 S V_1^T = \underset{m\times m}{U} \ \underset{m\times n}{\Sigma} \ \underset{n\times n}{V^T}, \quad \text{where } \Sigma = \underbrace{\left[\begin{array}{c|c} S & 0 \\ \hline 0 & 0 \end{array}\right]}_{m\times n} = \left[\begin{array}{c|c} \begin{matrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{matrix} & 0_{m\times(n-r)} \\ \hline 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{array}\right]_{m \times n}$$

The advantage here is that U & V are square & orthonormal.

Special cases:

- A is wide, $r = m < n$: independent rows $\rightarrow$ $\boxed{A} = \boxed{U}\ \underbrace{\left[\begin{array}{c|c} S & 0 \end{array}\right]}_{\Sigma}\ \boxed{V^T}$
- A is tall, $r = m > n$: independent columns $\rightarrow$ $\boxed{A} = \boxed{U}\ \underbrace{\left[\begin{array}{c} S \\ \hline 0 \end{array}\right]}_{\Sigma}\ \boxed{V^T}$

GEOMETRY OF SVD

$U^TU = I \qquad \|U\vec{x}\|^2 = \vec{x}^T U^T U \vec{x} = \vec{x}^T\vec{x} = \|\vec{x}\|^2$ U could be a rotation matrix!

$V^T\vec{x}$ reorients $\vec{x}$ with the same length

$\Sigma(V^T\vec{x})$ stretches $V^T\vec{x}$ along each axis by respective σ

$U(\Sigma V^T\vec{x})$ reorients with the same length

If $\|\vec{x}\| = 1$, $\|A\vec{x}\| \le \sigma_1 \longrightarrow \|A\vec{x}\| \le \sigma_1 \|\vec{x}\|$.

We assumed that A is real & claimed A^TA & AA^T have real eigenstuff

$(A^TA)^T = A^T(A^T)^T = A^TA \qquad (AA^T)^T = (A^T)^T A^T = AA^T$

CLAIM Symmetric $Q^T = Q$. Let $Q\vec{x} = \lambda\vec{x}$, $\lambda = a + jb$. $\bar{\lambda} = a - jb \rightarrow Q\bar{\vec{x}} = \bar{\lambda}\bar{\vec{x}}$. Show $b = 0$.

PROOF $\bar{\vec{x}}^T Q^T = \bar{\lambda}\bar{\vec{x}}^T \rightarrow \bar{\vec{x}}^T Q = \bar{\lambda}\bar{\vec{x}}^T \rightarrow \bar{\vec{x}}^T Q \vec{x} = \lambda \bar{\vec{x}}^T\vec{x} \rightarrow \bar{\vec{x}}^T Q\vec{x} = \bar{\lambda}\bar{\vec{x}}^T x$

$\hookrightarrow \lambda\bar{\vec{x}}^T\vec{x} = \bar{\lambda}\bar{\vec{x}}^T\vec{x} \rightarrow \lambda = \bar{\lambda}$

$\hookrightarrow a + jb = a - jb \rightarrow b = 0 \rightarrow \lambda$ is real!