EECS 16B ## 3.19 LECTURE 18 ##

FINDING AN SVD

Construct A^TA or AA^T as intermediaries. $A^TA \in \mathbb{R}^{n\times n}$ $AA^T \in \mathbb{R}^{m \times m}$

Claim (for after break): A^TA & AA^T have only real eigenvalues, and r of them are positive (rest are 0). Both also have a complete set of orthonormal eigenvectors.

PROCEDURE W/ A^TA

1. Find eigenvalues λ_i of A^TA in order largest to smallest, so that $\lambda_1 \geq \lambda_2 \geq \dots \lambda_r > 0 = \lambda_{r+1}, \dots, \lambda_n$
2. Find orthonormal eigenvectors $\vec{v}_1, \dots, \vec{v}_r$ corresponding to $\lambda_1, \dots, \lambda_r$
   $A^TA\vec{v}_i = \lambda_i \vec{v}_i$
3. Let $\sigma_i = \sqrt{\lambda_i}$ and obtain $\vec{u}_i$ from $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$

Given $\vec{v}_1, \dots, \vec{v}_r$, orthonormal as in Step 2, show that $\vec{u}_1, \dots, \vec{u}_r$ are also orthonormal, and that $\sigma_1 \vec{u}_1 \vec{v}_1^T + \dots + \sigma_r \vec{u}_r \vec{v}_r^T = A$.

1. We know $\vec{v}_1, \dots, \vec{v}_r$ satisfy $\vec{v}_j^T \vec{v}_i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$. Then, we say

$$\vec{u}_j^T \vec{u}_i = \frac{1}{\sigma_j}(A\vec{v}_j)^T \frac{1}{\sigma_i} A\vec{v}_i = \frac{1}{\sigma_j \sigma_i} \vec{v}_j^T A^T A \vec{v}_i = \frac{1}{\sigma_j \sigma_i} \lambda_i \vec{v}_j^T \vec{v}_i = \begin{cases} \dots \end{cases} \times \frac{1}{\sigma_j \sigma_i}\lambda_i$$

If $i = j$, $\frac{1}{\sigma_i \sigma_i}\lambda_i = \frac{1}{\sigma_i^2}\lambda_i = \frac{1}{\lambda_i}\lambda_i = 1$. ▨

2. $A\vec{v}_i = \sigma_i \vec{u}_i \longrightarrow A\underbrace{[\vec{v}_1 \cdots \vec{v}_r]}_{V_1} = [\sigma_1 \vec{u}_1 \cdots \sigma_r \vec{u}_r] = [\vec{u}_1 \cdots \vec{u}_r]\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}$

$$AV_1 = [\vec{u}_1 \cdots \vec{u}_r]\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \qquad AV_1V_1^T = [\vec{u}_1 \cdots \vec{u}_r]\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}\begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_r^T \end{bmatrix}$$

$$V_1V_1^T = [\vec{v}_1 \cdots \vec{v}_r]\begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_r^T \end{bmatrix} = I \qquad AV_1V_1^T = \vec{u}_1 \sigma_1 \vec{v}_1^T + \dots + \vec{u}_r \sigma_r \vec{v}_r^T = AI = A$$

You can also go the long way, with $V_2 = [\vec{v}_{r+1} \cdots \vec{v}_n]$, knowing that $A[V_1\ V_2] = I$, showing that $AV_2V_2^T = 0$, meaning that $AV_1V_1^T = I$.

PROCEDURE W/ AA^T

want this if $m < n$, or whichever is diagonal

1. Find eigenvalues of AA^T: $\lambda_1 \geq \dots \geq \lambda_r > 0 = \lambda_{r+1}, \dots, \lambda_m$
2. Find orthonormal eigenvectors: $AA^T\vec{u}_i = \lambda_i \vec{u}_i$
3. Let $\sigma_i = \sqrt{\lambda_i}$ and obtain $\vec{v}_i$ from $\vec{v}_i = \frac{1}{\sigma_i}A^T\vec{u}_i$

NOTE The methods of determining $\vec{u}_i$ & $\vec{v}_i$ are different for A^TA & AA^T!

EX $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$ $AA^T = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$ $A^TA = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$

diagonal, easier

1. $\lambda_1 = 32, \ \lambda_2 = 18$ 2. $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

3. $\sigma_1 = 4\sqrt{2}, \ \sigma_2 = 3\sqrt{2} \longrightarrow \vec{v}_1 = \frac{1}{4\sqrt{2}}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \vec{v}_2 = \frac{1}{3\sqrt{2}}\begin{bmatrix} -3 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$A = 4\sqrt{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} + 3\sqrt{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \end{bmatrix}$