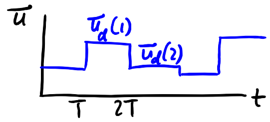
EECS 16B ## 3.10 Lecture 15

Discretization

$\vec{u}(t) \rightarrow \boxed{\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)} \xrightarrow{\vec{x}(t)} \underset{\text{"sampling"}}{\underset{T\ T}{\boxed{C/D}}} \rightarrow \vec{x}_d(k) = \vec{x}(kT)$



$\vec{x}_d(k+1) = A_d\vec{x}_d(k) + B_d\vec{u}_d(k)$

Scalar inputs: $x(t) = e^{\lambda(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)} b u(\tau)\, d\tau$

Let $t_0 = kT$, $t = (k+1)T$: $\underbrace{x((k+1)T)}_{x_d(k+1)} = \underbrace{e^{\lambda T}}_{\lambda_d}\underbrace{x(kT)}_{x_d(k)} + \underbrace{\int_{kT}^{kT+T} e^{\lambda(\overbrace{kT+T-\tau}^{s})} d\tau\, b u_d(k)}_{b_d = \int_0^T e^{\lambda s} ds\, b}$ $= \begin{cases} T & \text{if } \lambda = 0 \\ \frac{e^{\lambda T}-1}{\lambda} & \text{if } \lambda \neq 0 \end{cases}$

Vector case: first assume A is diagonal & $B = \vec{b}$ (single input)

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\vec{x}(t) + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}\vec{u}(t)$$

$\rightarrow \frac{d}{dt}x_i(t) = \lambda_i x_i(t) + b_i u(t)$ for all $i \in \{1 \ldots n\}$

$x_{d,i}(k+1) = e^{\lambda_i T}x_{d,i}(k) + \left(\int_0^T e^{\lambda_i s} ds\right) b_i u_d(k)$

$$\rightarrow \vec{x}_d(k+1) = \begin{bmatrix} e^{\lambda_1 T} & & \\ & \ddots & \\ & & e^{\lambda_n T} \end{bmatrix}\vec{x}_d(k) + \begin{bmatrix} \int e^{\lambda_1 s} ds & & \\ & \ddots & \\ & & \int e^{\lambda_n s} ds \end{bmatrix} u_d(k)$$

now assume A is diagonalizable, but <u>not</u> diagonal

$V = [\vec{v}_1 \ldots \vec{v}_n]$ where $\vec{v}_i$ is independent, so V^{-1} exists. $\vec{z} = V^{-1}\vec{x}$

1. Change variables to $\vec{z}$.
2. $\frac{d}{dt}\vec{z}(t) = \Lambda\vec{z}(t) + V^{-1}B u(t)$
3. Discretize system with z.
4. Go back to $\vec{x} = V\vec{z}$: $\vec{x}_d(k) = V\vec{z}_d(k) \longrightarrow \vec{x}_d(k+1) = V\vec{z}_d(k+1)$
5. $\vec{x}_d(k+1) = \underbrace{V\begin{bmatrix} e^{\lambda_1 T} & & \\ & \ddots & \\ & & e^{\lambda_n T} \end{bmatrix}V^{-1}}_{A_d}\vec{x}_d(k) + \underbrace{V\begin{bmatrix} \int e^{\lambda_1 s} ds & & \\ & \ddots & \\ & & \int e^{\lambda_n s} ds \end{bmatrix}V^{-1}B}_{B_d} u_d(k)$

<u>EX.</u> $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad \frac{d}{dt}\vec{x} = A\vec{x} \longrightarrow \vec{x}_d(k+1) = A_d\vec{x}_d(k)$ $\quad V = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix}$ $\quad V^{-1} = \frac{1}{2j}\begin{bmatrix} j & -1 \\ j & 1 \end{bmatrix}$

$\lambda_1 = j \quad \lambda_2 = -j$

Then, $A_d = V\begin{bmatrix} e^{jT} & \\ & e^{-jT} \end{bmatrix}V^{-1} = \begin{bmatrix} \frac{1}{2}(e^{jT}+e^{-jT}) & -\frac{1}{2j}(e^{jT}-e^{-jT}) \\ \frac{1}{2j}(e^{jT}-e^{-jT}) & \frac{1}{2}(e^{jT}+e^{-jT}) \end{bmatrix} = \begin{bmatrix} \cos T & -\sin T \\ \sin T & \cos T \end{bmatrix}$

Remember the scalar formula!

CONTROLLABILITY

$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$

if given $\vec{x}(0)$ & $\vec{u}(0), \vec{u}(1), \vec{u}(2), \ldots,$ what's the solution?

$\vec{x}(1) = A\vec{x}(0) + B\vec{u}(0)$ $\quad$ $\vec{x}(2) = A^2\vec{x}(0) + AB\vec{u}(0) + B\vec{u}(1)$

$\vec{x}(3) = A^3\vec{x}(0) + A^2B\vec{u}(0) + AB\vec{u}(1) + B\vec{u}(2)$

$\vec{x}(k) = A^k\vec{x}(0) + \left[A^{k-1}B \mid \cdots \mid AB \mid B\right]\begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}$

assume $B = \vec{b}$: $\quad$ $\vec{x}(k) - A^k\vec{x}(0) = \left[\vec{b} \;\; A\vec{b} \;\; \cdots \;\; A^{t-1}\vec{b}\right]\begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}$

can we reach any desired state $\vec{x}_{target}$ at some t from $\vec{x}(0)$ with an appropriate input sequence $u(0), u(1), \ldots, u(t-1)$?

if yes, the system is called controllable; $\text{span}\{\vec{b}, A\vec{b}, \ldots, A^{t-1}\vec{b}\} = \mathbb{R}^n$

EX $\quad A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad n = 2$

$\text{span}\{\vec{b}, A\vec{b}\} = \left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \mathbb{R}^2$

controllable!

EX $\quad A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\text{span}\{\vec{b}, A\vec{b}, A^2\vec{b}\} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

not controllable!

for any $\mathbb{R}^k$