EECS 16B ## 3.3 LECTURE 13

$\frac{d}{dt}x(t) = f(x(t), u(t))$ $\qquad x(t), u(t) \in \mathbb{R}$

$f(x^*, u^*)$ $\qquad$ equilibrium condition

Taylor Approximation

$$f(x,u) = \underbrace{f(x^*, u^*)}_{0} + \underbrace{\partial f/\partial x \big|_{x^*, u^*}}_{\lambda} \underbrace{(x-x^*)}_{\tilde{x}} + \underbrace{\partial f/\partial u \big|_{x^*, u^*}}_{b} \underbrace{(u-u^*)}_{\tilde{u}}$$

$\tilde{x}, \tilde{u}$ are perturbation variables $\qquad f(x,u) \approx \lambda \tilde{x} + b\tilde{u}$

$\frac{d}{dt}\tilde{x}(t) = \frac{d}{dt}(x(t) - x^*) = \frac{d}{dt}x(t)$ $\qquad$ right side should only be in $\tilde{x}$

$= f(x,u) = f(x^* + \tilde{x}, u^* + \tilde{u})$ $\qquad$ and $\tilde{u}$

$\frac{d}{dt}\tilde{x}(t) = \lambda \tilde{x}(t) + b\tilde{u}(t)$

EX Say you have a car mass m velocity $v(t)$. We can apply $u(t)$ wheel torque. Drag $\frac{1}{2}\rho A c (v(t))^2$ will slow us down. We know $F = ma$.

$m \, dv/dt = -\frac{1}{2}\rho A c v^2 + \frac{1}{R} u$ $\qquad$ where R is the wheel radius.

for equilibrium, want v^*. what should u^* be?

$u^* = \frac{1}{2}\rho A c (v^*)^2 R$ $\qquad$ where u counters drag force. let's linearize.

$dv/dt = f(v,u) = -\frac{1}{2M}\rho A c v^2 + \frac{1}{RM} u$ $\qquad \lambda = \partial f/\partial v \big|_{eq} = -\frac{1}{M}\rho A c v^*$

$\qquad b = \partial f/\partial u \big|_{eq} = \frac{1}{RM}$

$d\tilde{v}/dt = \lambda \tilde{v} + b\tilde{u}$ $\qquad$ linearized! cruise control uses this!

if $\tilde{u}(t) = 0 \ \forall t$, $d\tilde{v}/dt = \lambda \tilde{v}$. given $\tilde{v}(t_0)$, $\tilde{v}(t) = \tilde{v}(t_0)e^{\lambda t}$

$\lambda < 0$, so as $t \to \infty$, $\tilde{v}(t) \to 0$. $\qquad$ if you perturb but the system returns to equilibrium, this eq. point is stable.

Vector Case

$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t), \vec{u}(t))$, $\quad f(x^*, u^*) = 0$, $\quad x^*$ & u^* are vectors

Taylor: $f(x,u) = \underbrace{f(x^*, u^*)}_{0} + \underbrace{\nabla_x f(\vec{x}, \vec{u})}_{A} \underbrace{(\vec{x} - x^*)}_{\tilde{\vec{x}}} + \underbrace{\nabla_u f(\vec{x}, \vec{u})}_{B} \underbrace{(\vec{u} - u^*)}_{\tilde{\vec{u}}}$

$$\nabla_x f(\vec{x}, \vec{u}) = \text{jacobian matrix} = \begin{bmatrix} \frac{\partial f_1(\vec{x},\vec{u})}{\partial x_1} & \frac{\partial f_1(\vec{x},\vec{u})}{\partial x_2} & \cdots & \frac{\partial f_1(\vec{x},\vec{u})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(\vec{x},\vec{u})}{\partial x_1} & \frac{\partial f_n(\vec{x},\vec{u})}{\partial x_2} & \cdots & \frac{\partial f_n(\vec{x},\vec{u})}{\partial x_n} \end{bmatrix}$$

$\nabla_u f(\vec{x}, \vec{u})$ = along the same line, through probably a column vector (1 input).

$\frac{d}{dt}\tilde{\vec{x}} = \frac{d}{dt}\vec{x} = f(x^* + \tilde{\vec{x}}(t), u^* + \tilde{\vec{u}}(t)) \approx A\tilde{\vec{x}}(t) + B\tilde{\vec{u}}(t)$ $\qquad$ linear!

Discrete-Time Systems

$\vec{x}(t+1) = f(\vec{x}(t), \vec{u}(t))$ linear if $f(\vec{x},\vec{u})$ can $= A\vec{x} + B\vec{u}$: $\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$

This is very functionally similar to continuous time systems.

EX. Warehouse: $s(t)$: inventory on morning of day t

$g(t)$: goods manufactured that day

input! $u_1(t)$: goods sold that day

$s(t+1) = s(t) + g(t) - u_1(t)$

$r(t)$: raw material available. assume 1 day to manufacture.

$g(t+1) = r(t)$

input! $u_2(t)$: amount of r ordered. assume same-day delivery.

$r(t+1) = u_2(t)$ everything is linear: $A\vec{x}(t) + B\vec{u}(t)$

$$\underbrace{\begin{bmatrix} s(t+1) \\ g(t+1) \\ r(t+1) \end{bmatrix}}_{\vec{x}(t+1)} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} s(t) \\ g(t) \\ r(t) \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{\vec{u}(t)}$$

Discrete Equilibrium

$\vec{x}^*$ is in equilibrium if $\vec{x}^* = f(\vec{x}^*, \vec{u}^*)$.

NOTE Difference from continuous, where $\hookrightarrow = 0$.