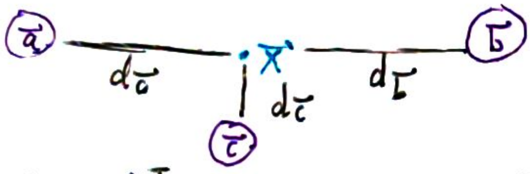
EECS 16A ## 11.19 LECTURE 23

Step 3: Converting distances to location

TRILATERATIONS

2D:



$\|\vec{x}-\vec{a}\|^2 = d_a^2 \longrightarrow 1$

$\|\vec{x}-\vec{b}\|^2 = d_b^2 \longrightarrow 2$

$\|\vec{x}-\vec{c}\|^2 = d_c^2 \longrightarrow 3$

$\longrightarrow 4$

1. $(\vec{x}-\vec{a})^T(\vec{x}-\vec{a}) = (\vec{x}^T-\vec{a}^T)(\vec{x}-\vec{a}) = \|\vec{x}\|^2 + \|\vec{a}\|^2 - 2\langle \vec{x}, \vec{a}\rangle = d_a^2$
2. $\|\vec{x}\|^2 + \|\vec{b}\|^2 - 2\langle \vec{x}, \vec{b}\rangle = d_b^2 \longrightarrow 5$
3. $\|\vec{x}\|^2 + \|\vec{c}\|^2 - 2\langle \vec{x}, \vec{c}\rangle = d_c^2 \longrightarrow 6$

4.-5. $\|\vec{a}\|^2 - \|\vec{b}\|^2 - 2\langle \vec{x}, \vec{a}\rangle + 2\langle \vec{x}, \vec{b}\rangle = d_a^2 - d_b^2$

$\|\vec{a}\|^2 - \|\vec{b}\|^2 - 2\vec{a}^T\vec{x} + 2\vec{b}^T\vec{x} = d_a^2 - d_b^2 \longrightarrow 7$

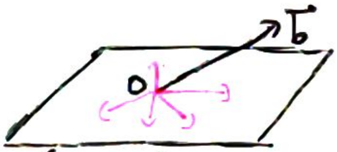
4.-6 $\|\vec{a}\|^2 - \|\vec{c}\|^2 - 2\vec{a}^T\vec{x} + 2\vec{c}^T\vec{x} = d_a^2 - d_c^2 \longrightarrow 8$

$$\begin{matrix}7.\\8.\end{matrix}\quad \begin{bmatrix} 2(\vec{b}^T-\vec{a}^T) \\ 2(\vec{c}^T-\vec{a}^T) \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} d_a^2 - d_b^2 - \|\vec{a}\|^2 + \|\vec{b}\|^2 \\ d_a^2 - d_c^2 - \|\vec{a}\|^2 + \|\vec{c}\|^2 \end{bmatrix}$$

But what about noise?

$A\vec{x} \approx \vec{b}$. Find $\vec{x}$ such that $A\vec{x}$ is as close to $\vec{b}$ as possible.

LEAST SQUARES



$A\vec{x} \in$ colspace(A)

Cases to consider: A is $m \times n$ $(m \neq n)$, more eqs than unks.

ex. $A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \vec{a}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

we can move span$\{\vec{a}\}$

THM shortest distance between a point & a line is given by the perpendicular from point to line

$\vec{z} = \alpha \cdot \vec{a}$ $\qquad \vec{z} \in$ span$\{\vec{a}\}$

say r is closer to b than q: $bq^2 + qr^2 = br^2$

hypotenuse is longest: $\underline{br > bq}$ & $br > qr$

$\therefore \vec{z}$ is closest. also, $\vec{z}$ is the projection of $\vec{b}$ onto $\vec{a}$.

But how do we find α? $\quad \vec{y} = qb = \vec{b} - \vec{z}$

$$\langle \vec{a}, \vec{y} \rangle = 0 \qquad \vec{a}^T\vec{b} - \vec{a}^T(\alpha \cdot \vec{a}) = 0$$

$$\vec{a}^T(\vec{b} - \vec{z}) = 0 \qquad \langle \vec{a}, \vec{b} \rangle - \alpha \|\vec{a}\|^2 = 0$$

DEF $\quad \alpha = \dfrac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\|^2}$ least squares projection

If $\vec{b} = -\vec{a}$, $\alpha = -1$. If $\vec{b} = \vec{a}$, $\alpha = 1$.

ORTHOGONALITY BEYOND 2D

THM Consider A and let $\vec{y} \in \text{colspace}(A)$.

$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$ Let $\vec{z}$ such that $\langle \vec{z}, \vec{a}_1 \rangle = \langle \vec{z}, \vec{a}_2 \rangle = \langle \vec{z}, \vec{a}_n \rangle = 0$

Then $\langle \vec{z}, \vec{y} \rangle = 0$

PROOF: $\vec{y} = c_1\vec{a}_1 + c_2\vec{a}_2 + \quad c_n\vec{a}_n$, since $\vec{y} \in \text{colspace}(A)$

$$\langle \vec{z}, \vec{y} \rangle = \langle \vec{z}, c_1\vec{a}_1 + \cdot\ c_n\vec{a}_n \rangle = \langle \vec{z}, c_1\vec{a}_1 \rangle + \quad \langle \vec{z}, c_n\vec{a}_n \rangle$$

$$= c_1\langle \vec{z}, \vec{a}_1 \rangle + c_2\langle \vec{z}, \vec{a}_2 \rangle + \quad c_n\langle \vec{z}, \vec{a}_n \rangle = 0 \quad \blacksquare$$

GENERALIZING LEAST SQUARES

$A\vec{x} \approx \vec{b}$ $\qquad$ A is $m \times n$: m equations, n unknowns

noisy!

Find $\vec{x}$ where $A\vec{x}$ is close to $\vec{b}$; $\|\vec{e}\|^2 = \|A\vec{x} - \vec{b}\|^2$

$A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$ Find the orthogonal projection of $\vec{b}$ onto $\text{col}(A)$. Call this $\vec{z}$, such that

$$\langle (\vec{b} - \vec{z}), \vec{a}_1 \rangle = \langle (\vec{b} - \vec{z}), \vec{a}_2 \rangle = \quad \langle (\vec{b} - \vec{z}), \vec{a}_n \rangle = 0$$

$$\vec{a}_1^T(\vec{b} - \vec{z}) = 0 \qquad \vec{a}_n^T(\vec{b} - \vec{z}) = 0 \qquad A^T = \begin{bmatrix} \text{---} & \vec{a}_1 & \text{---} \\ & \vdots & \\ \text{---} & \vec{a}_n & \text{---} \end{bmatrix}$$

$$A^T(\vec{b} - \vec{z}) = 0 \qquad A^T\vec{b} - \underbrace{A^TA}_{\text{square}}\vec{x} = 0$$

$$A^T\vec{b} = A^TA\vec{x}$$

DEF $\quad \vec{x} = (A^TA)^{-1}A^T\vec{b}$ $\qquad$ general least squares algorithm projection of $\vec{b}$ onto colspace(A)