EECS 16A ## 10.1 LECTURE 10

PROPERTIES OF EIGENVALUES/VECTORS

Q1: What if we initialize with an arbitrary $\vec{x}(0)$?

Q2: How do we improve imaging performance?

THEOREM $A = n \times n$ matrix. $\lambda_1, \lambda_2, .. \lambda_n$ are e-vals of A.
$\lambda_i \neq \lambda_j$ for all i, j. $\vec{v}_1, \vec{v}_2, .. \vec{v}_n$ e-vec.

If all e-vals are different, then $\vec{v}_1, \vec{v}_2, .. \vec{v}_n$ is basis $(\mathbb{R}^n)$

PROOF For A, a 2x2 matrix

Show. 1. $\vec{v}_1, \vec{v}_2$ indep.
2. $\vec{v}_1, \vec{v}_2 \in \text{span}(\mathbb{R}^2)$

λ_1, λ_2 $\quad \lambda_1 \neq \lambda_2 \quad A\vec{v}_1 = \lambda_1 \vec{v}_1$

$\hookrightarrow \vec{v}_1 \quad \hookrightarrow \vec{v}_2 \quad A\vec{v}_2 = \lambda_2 \vec{v}_2$

If possible, assume that $\vec{v}_1$ & $\vec{v}_2$ are linearly dependent

$\vec{v}_1 = \alpha \vec{v}_2$, $\alpha \neq 0$ $\quad A\vec{v}_1 = A\alpha\vec{v}_2 = \alpha\lambda_2\vec{v}_2 = \lambda_1\vec{v}_1 = \lambda_1 \alpha \vec{v}_2$

$\hookrightarrow \alpha\lambda_2\vec{v}_2 = \alpha\lambda_1\vec{v}_2 \;\lightning$

We were given that $\lambda_1 \neq \lambda_2$, so this is a contradiction. Thus, $\vec{v}_1$ & $\vec{v}_2$ are independent, thus spanning $\mathbb{R}^2$, thus being a basis for $\mathbb{R}^2$. $\blacksquare$ QED.

GENERAL INITIAL STATES Q1

$\vec{x}(t+1) = A \cdot \vec{x}(t)$ $\quad$ Dynamic System

If $\vec{x}(0)$ is steady, the state doesn't change

If $\vec{x} \to$ arbitrary, write $\vec{x}$ as a combination of e-vectors

$\vec{x} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + .\, \alpha_n\vec{v}_n$ $\quad$ where λ_i's $\to$ distinct

$\hookrightarrow A\vec{x} = A(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + .\, \alpha_n\vec{v}_n)$

$= (\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 + .\, \lambda_n\vec{v}_n)\cdot\alpha_i = \alpha_1\lambda_1\vec{v}_1 + \alpha_2\lambda_2\vec{v}_2 + \alpha_n\lambda_n\vec{v}_n$

$A^2\vec{x} = \alpha_1\lambda_1^2\vec{v}_1 + \ldots \alpha_n\lambda_n^2\vec{v}_n$

$A^t\vec{x} = \alpha_1\lambda_1^t\vec{v}_1 + .. \alpha_n\lambda_n^t\vec{v}_n \longrightarrow$

$\lambda_i = 1$	no change
$\lambda_i > 1$	explode
$\lambda_i < 0$	transient/will die

EX $Q = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \downarrow (\frac{3}{4} - \lambda)(\frac{1}{2} - \lambda) - \frac{1}{8}$

$\lambda^2 - \frac{5}{4}\lambda + \frac{1}{4} \to \lambda = 1, \frac{1}{4}$

$1: \begin{bmatrix} -1/4 & 1/2 \\ 1/4 & -1/2 \end{bmatrix} \to \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{v}_1$

$\frac{1}{4}: \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 1/4 \end{bmatrix} \to \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \vec{v}_2$

$Q^t\vec{x} = \alpha_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha_2 \cdot \frac{1}{4^t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lim_{t\to\infty} Q^t\vec{x} = \alpha_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Indistinct Eigenvalues

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\lambda_1 = 1$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{v}_1, \vec{v}_2$ are still a basis $(\mathbb{R}^2)$

$\lambda_2 = 1$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\lambda_1 = 0$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ eigenspace dimension is 1.

$\lambda_2 = 0$ $\vec{v}_2$ doesn't exist. not a basis $(\mathbb{R}^2)$

This is out of scope for 16A Covered in 16B.

Eigenvalue 0

$A\vec{v} = 0\vec{v} = 0$ $\vec{v}$ is in Null(A)

Columns of A are linearly dependent

A is not invertible & $A\vec{x} = \vec{b} \rightarrow$ no sol.

Small eigenvalues make inversion challenging.

For more, take 16B, 126, 189

Eigenvalues in Imaging

$\vec{s} = H\vec{i} + \vec{\omega}$ (noise)

$H^{-1}\vec{s} = H^{-1}H\vec{i} + H^{-1}\vec{\omega} = \vec{i} + \underline{H^{-1}\vec{\omega}}$

we want to minimize this

For H^{-1} to have small e-vals, H needs large e-vals.

Recap: Module 1 – Intro to Systems

Imaging Lab, Tomography $\longleftrightarrow$ Systems of Lin. Eq., Gaussian Elimination $\longleftrightarrow$ Matrices, Vectors, Linear Transforms.

Matrices, Vectors, Linear Transforms. $\updownarrow$ matrix mult. Inversion, Linear (In)dependence

Steady States for Dynamical Sys $\longleftrightarrow$ Vectorspace, Columnspace, Null space, Eigen space $\longleftrightarrow$ Inversion, Linear (In)dependence

Steady States for Dynamical Sys $\updownarrow$ Better Imaging Imaging Lab, Tomography