EECS 16A ## 9.26 LECTURE 9

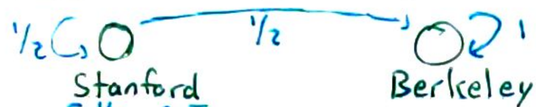
Recap

Determinant $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

if $ad - bc = 0 \rightarrow ad = bc \rightarrow \frac{a}{c} = \frac{b}{d} = \frac{1}{\alpha} \rightarrow \begin{matrix} c = \alpha a \\ d = \alpha b \end{matrix}$

$\hookrightarrow \begin{bmatrix} a & b \\ \alpha a & \alpha b \end{bmatrix}$

Tale of 2 Websites



$Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$

$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{x}[t] = \begin{bmatrix} (1/2)^t \\ 1-(1/2)^t \end{bmatrix}$

$\vec{x}[t+1] = Q \cdot \vec{x}[t]$

$t \rightarrow \infty \quad \vec{x}[t] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

What if we start at $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

$\begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x}_{steady} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a vector such that $Q\vec{x}_{steady} = \vec{x}_{steady}$

General Problem

$Q\vec{x} = \vec{x} \longrightarrow Q\vec{x} - \vec{x} = \vec{0} \longrightarrow (Q - I)\vec{x} = \vec{0}$ $\quad \vec{x}$ is on invariate direct.

All $\vec{x} \in \text{Null}(Q - I)$ satisfy this.

Going Back

$\begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 \\ 1/2 & 0 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{matrix} \rightarrow x_1 = 0 \\ \rightarrow x_2 = t \end{matrix}$

$\text{Null}(Q - I) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

Eigenvalues, Eigenvectors, Eigenspaces

$Q\vec{x} = \lambda\vec{x} \qquad \lambda \in \mathbb{R}$

DEF λ is an eigenvalue of Q.

DEF $\vec{x}$ is an eigenvector of Q corresponding to eigenvalue λ.

DEF $\vec{x}$ belongs to the eigenspace corresponding to eigenvalue λ.

ex. $Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$ find $\lambda, \vec{x}$ such that $Q\vec{x} = \lambda\vec{x}$

$Q\vec{x} - \lambda I\vec{x} = 0$ $\quad$ when is $\text{null}(Q - \lambda I)$ non-trivial? ie. $> \{\vec{0}\}$

$(Q - \underbrace{\lambda}_{\text{unknown}} I)\underbrace{\vec{x}}_{\text{unknown}} = 0$ $\quad$ use the determinant! $\begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1/2 - \lambda & 0 \\ 1/2 & 1 - \lambda \end{bmatrix}$

$D = (1/2 - \lambda)(1 - \lambda) - (0) = 1/2 - 3/2\,\lambda + \lambda^2 = 0$

$\lambda_1 = 1/2$, $\lambda_2 = 1$ are the two solutions.

continued on next page

$\lambda_2 = 1$ is an eigenvalue

$\text{Null}(Q-\lambda_2 I) = \text{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$, the corresponding eigenspace

DEF If eigenvalue = 1, the vectors in the eigenspace are in a "steady state"

$\lambda_1 = 1/2 \rightarrow Q-\lambda_1 I = \begin{bmatrix}0 & 0\\ 1/2 & 1/2\end{bmatrix}$ $\quad \text{Null} = \left[\begin{array}{cc|c}0 & 0 & 0\\ 1/2 & 1/2 & 0\end{array}\right] \rightarrow \left[\begin{array}{cc|c}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right] \rightarrow x_1+x_2=0$

$x_1 = -x_2$

$x_2 = t$

Vectors $\in \text{Null}(Q-\lambda_1 I) = \text{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$

$\vec{\varpi} = \begin{bmatrix}-1\\1\end{bmatrix} \longrightarrow Q\vec{\varpi} = \frac{1}{2}\vec{\varpi}$ $\qquad \begin{bmatrix}1/2 & 0\\ 1/2 & 1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\end{bmatrix} = \frac{1}{2}\vec{\varpi}$

$Q(Q\vec{\varpi}) = Q\left(\frac{1}{2}\vec{\varpi}\right) = \frac{1}{4}\vec{\varpi}$

$Q^t\vec{\varpi} = \left(\frac{1}{2}\right)^t\vec{\varpi} = \lambda^t\vec{\varpi}$

DEF If, as in the example above, the eigenvalue <1, the vectors are "transient."

ex. $A = \begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$

1. Consider $(A-\lambda I) = \begin{bmatrix}1-\lambda & 2\\ 4 & 3-\lambda\end{bmatrix}$
2. $\text{Det}(A-\lambda I) = (1-\lambda)(3-\lambda) - 8$
3. Set det = 0 to find eigenvalues

   $\lambda^2 - 4\lambda - 5 = 0 \rightarrow \lambda_1 = 5,\ \lambda_2 = -1$
4. Find $\text{Null}(A-5I)$ $\qquad A-5I = \begin{bmatrix}-4 & 2\\ 4 & -2\end{bmatrix}$

   $\left[\begin{array}{cc|c}-4 & 2 & 0\\ 4 & -2 & 0\end{array}\right] \rightarrow \left[\begin{array}{cc|c}1 & -1/2 & 0\\ 0 & 0 & 0\end{array}\right]$ $\quad x_1 - x_2/2 = 0 \quad x_1 = \frac{1}{2}x_2$

   $x_2 = t$

   $\text{Null}(A-5I) = \text{span}\left\{\begin{bmatrix}1/2\\1\end{bmatrix}\right\}$ = eigenspace cor. 5

   Any vector in is an eigenvector cor. to 5

$\vec{v} = \begin{bmatrix}1/2\\1\end{bmatrix}$

$A\vec{v} = 5\vec{v}$

$A^t\vec{v} = 5^t\vec{v}$

DEF If eigenvalue >1, the vectors are "blowing up."

5. Find $\text{Null}(A+I)$ $\qquad A+I = \begin{bmatrix}2 & 2\\ 4 & 4\end{bmatrix}$

   $\left[\begin{array}{cc|c}2 & 2 & 0\\ 4 & 4 & 0\end{array}\right] \rightarrow \left[\begin{array}{cc|c}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right]$ $\quad x_1 + x_2 = 0 \quad x_1 = -x_2$

   $x_2 = t$

   $\text{Null}(A+I) = \text{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ = eigenspace cor. -1

   Any vector in is an eigenvector cor. to -1

$\vec{y} = \begin{bmatrix}-1\\1\end{bmatrix}$

$A\vec{y} = -\vec{y}$

$A^t\vec{y} = (-1)^t\vec{y}$