EECS 16A ## 9.19 LECTURE 7

THEOREM: If A is an invertible matrix, then $A\vec{x}=\vec{b}$ has a unique solution.

PROOF: We know that A^{-1} exists Consider $\vec{x}_0 = A^{-1}\cdot\vec{b}$.

$A\cdot\vec{x}_0 = A(A^{-1}\cdot\vec{b}) = I\cdot\vec{b} = \vec{b}$.

There is only one solution, meaning all solutions $=\vec{x}_0$.

Let $\vec{x}_1$ be another solution.

$A\cdot\vec{x}_1 = \vec{b} \longrightarrow A^{-1}\cdot A\cdot\vec{x}_1 = A^{-1}\cdot\vec{b} \longrightarrow I\vec{x}_1 = \vec{x}_0 \longrightarrow \vec{x}_1 = \vec{x}_0$.

QED.

If A is Invertible, $A\vec{x}=\vec{b}$ has a unique solution, AND A has linearly dependent columns.] A is square

To prove the other way around requires 16B complexity.

If A is invertible, A has a trivial nullspace.

NULLSPACES

$A\vec{x}=\vec{0}$ How many solutions?

$\hookrightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \longrightarrow$ unique solution!

$\hookrightarrow A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \longrightarrow$ infinite solutions!

If $A\vec{x}=\vec{0}$ has a solution $\vec{x}_0 \neq 0$, then $\alpha\vec{x}_0$ is also a solution $\alpha \in \mathbb{R} \longrightarrow \text{span}\{\vec{x}_0\}$ is a solution.

A does not need to be a square.

DEF The set of all solutions to $A\vec{x}=\vec{0}$ is called A's Nullspace.

Nullspaces can help characterize solutions to $A\vec{x}=\vec{b}$.

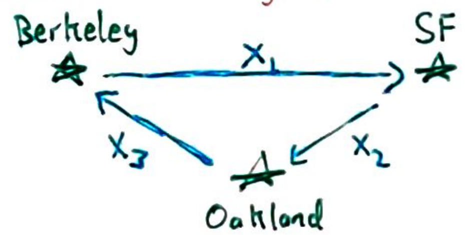
Say $\vec{x}_0$ is a solution to $A\vec{x}=\vec{0}$ (homogeneous eq.)

$\vec{x}_1$ is a solution to $A\vec{x}=\vec{b}$ (particular eq.)

$\vec{v} = \vec{x}_1 + \alpha\vec{x}_0$ is also a solution to $A\vec{x}=\vec{b}$

$\hookrightarrow$ parameter

EXAMPLE: Modeling Traffic Flows



How many measurements are needed to find x_1, x_2, x_3?

$$\begin{aligned} x_1 - x_2 &= 0 \\ x_2 - x_3 &= 0 \\ x_3 - x_1 &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

Assume no stopping/accumulation.

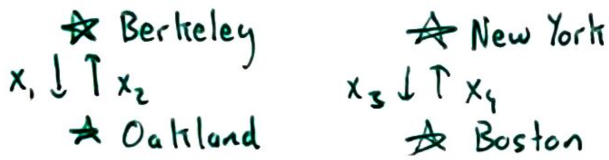
$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right] \begin{array}{l} \longrightarrow [\,1\ 0\ {-1}\,|\,0\,] \rightarrow x_1 = x_3 = t \\ \longrightarrow \quad \longrightarrow \quad \longrightarrow x_2 = x_3 = t \\ \longrightarrow x_3 \text{ is free} \rightarrow x_3 = t \end{array}$$

no leading 1.

All vectors $\vec{x} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$ are solutions $\longrightarrow$ span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ = Nullspace(A)

Only 1 variable is free, so only 1 measurement is needed.

Ex 2



$$\left[\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

2 free variables → dimension of Nspace

VECTORSPACE

See <u>Lecture 7 Notes</u> at eecs16a.org.
attached to PDF!

Pay special attention to the closure identities.

Say $(\mathbb{V}, \mathbb{F})$: set of independent $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$
all $\vec{v} \in \mathbb{V}$ are in span$\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$] BASIS

DEF Basis is the minimum spanning set.

DEF Dimension of Vs is # of elements in basis

Vector Spaces Addendum

This addendum is a summarized version of Section 7.1 of Note 7.

A **vector space** $\mathbb{V}$ is a set of vectors and two operators that satisfy the following properties:

- **Closure Under Vector Addition**: For any two vectors $\vec{v}, \vec{u} \in \mathbb{V}$, their sum $\vec{v} + \vec{u}$ must also be in $\mathbb{V}$.
- **Closure Under Scalar Multiplication**: For any vector $\vec{v} \in \mathbb{V}$ and scalar $\alpha \in \mathbb{R}$, the product $\alpha\vec{v}$ must also be in $\mathbb{V}$.
- **Contain the Zero Vector**: The vector $\vec{0}$ must be contained inside $\mathbb{V}$.

We have already dealt with vector spaces! When we write $\mathbb{R}^n$, we are describing the vector space of all n-dimensional vectors. If we applied the vector addition and scalar multiplication operations learned in previous lectures, we'd see that $\mathbb{R}^n$ satisfies all of the conditions we just mentioned above.

The set of all matrices is also a vector space $\mathbb{R}^{n \times m}$, but EECS16A generally deals only with $\mathbb{R}^n$.