EECS 16A

9.12 LECTURE 5

MORE PROOFS

"Without loss of generality" WLOG, where the essence of the original problem is not lost when using a simpler example.

ex. 3 numbers a, b, c

WLOG, let a be the largest, c be the smallest

This will normally be in relabeling/reordering cases.

THEOREM: If $A\vec{x} = \vec{b}$ has 2 or more solutions, then the columns of A are linearly dependent.

PROOF: Let $\vec{x}_1$ & $\vec{x}_2$ be 2 distinct solutions.

$$A\vec{x}_1 = \vec{b} \quad \text{and} \quad A\vec{x}_2 = \vec{b}$$

SHOW: For some i, $\vec{a}_i = \sum_{j=1, j\neq i}^{n} c_j \cdot \vec{a}_j$

$$A\vec{x}_1 = A\vec{x}_2 \longrightarrow A\underbrace{(\vec{x}_1 - \vec{x}_2)}_{\text{let } = \vec{y} \neq 0 \text{ (distinct!)}} = 0$$

$$A\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = 0 \longrightarrow y_1 \cdot \vec{a}_1 + y_2 \cdot \vec{a}_2 + \dots y_n \vec{a}_n = 0$$

At least one y_i has to be nonzero

WLOG, let $y_1 \neq 0 \longrightarrow \vec{a}_1 = -\frac{y_2}{y_1} \cdot \vec{a}_2 + \dots + -\frac{y_n}{y_1} \cdot \vec{a}_n$

QED.

LINEAR DEPENDENCE

Second Definition: $\{\vec{a}_1, \vec{a}_2, \dots \vec{a}_n\}$ are linearly dependent if there exist c_1, c_2, c_n, not all zero, such that $c_1 \cdot \vec{a}_1 + c_2 \cdot \vec{a}_2 + \quad c_n \vec{a}_n = 0$.

LINEAR TRANSFORMATIONS

Matrices are linear operators.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Rotation Matrix — Reflection $\left(\begin{smallmatrix} x \\ y \end{smallmatrix} \to \begin{smallmatrix} x \\ -y \end{smallmatrix}\right)$

Matrix is a linear trans.

f is a linear trans. if:

$f(x, y) = f(x) + f(y)$ and

$f(\alpha x) = \alpha f(x)$

eg. $f(\vec{x}) = 2\vec{x}$

$f(\vec{x} + \vec{y}) = 2\vec{x} + 2\vec{y} = f(\vec{x}) + f(\vec{y})$

$f(\alpha\vec{x}) = 2\alpha\vec{x} = \alpha f(\vec{x})$

STATE TRANSFORMATIONS

Vectors often represent the state of a system.

eg. state of car $\to$ $x(t), y(t), v(t) \to \vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \\ v(t) \end{bmatrix}$

EECS 16B dives into this with Control Theory

eg. System of Pumps & Reservoirs

(run pump)

$x_A(t)$ = water in tank A @ time t.

What is $\vec{x}(t+1)$?

$x_C(t+1) = x_B(t)$ | $x_A(t+1) = x_A(t)$ | $x_B(t+1) = x_C(t)$

$$\underbrace{\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix}}_{\vec{x}(t+1)} = \underbrace{\begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}}_{\vec{x}(t)} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to Q$$

State Transition Matrix

What about $\vec{x}(t+2)$ in terms of $\vec{x}(t)$? $\vec{x}(t+2000)$?

$Q\,\vec{x}(t) = \vec{x}(t+1) \qquad Q\vec{x}(t+1) = \vec{x}(t+2)$

$$\boxed{Q \cdot Q} \cdot \vec{x}(t) = \vec{x}(t+2)$$

See next page for information

$$Q \cdot Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

identity trans.

MATRIX-MATRIX MULTIPLICATION

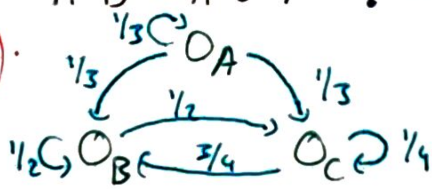
stacked matrix-vector mult!

$$A \cdot B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \underbrace{\boxed{\begin{matrix} b_{11} \\ b_{21} \end{matrix}}}_{\vec{b}_1} & \underbrace{\boxed{\begin{matrix} b_{12} \\ b_{22} \end{matrix}}}_{\vec{b}_2} \end{bmatrix} = A[\vec{b}_1 \ \ \vec{b}_2] = \begin{bmatrix} | & | \\ A\vec{b}_1 & A\vec{b}_2 \\ | & | \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$A \cdot B = A[\vec{b}_1 \ \ \vec{b}_2 \ \ldots \ \vec{b}_n] = [A\vec{b}_1 \ \ A\vec{b}_2 \ \ldots \ A\vec{b}_n]$$

eg.



$$Q = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{bmatrix} 1/3 & 0 & 0 \\ 1/3 & 1/2 & 3/4 \\ 1/3 & 1/2 & 1/4 \end{bmatrix} \begin{array}{l} \text{inflow to } A \\ B \\ C \end{array}$$

$$\begin{array}{ccc} \text{outflow of } A & B & C \end{array}$$

$Q \cdot Q \cdot Q \cdots Q$ to see water levels at the end of time

np.matmul(Q, Q)   np.linalg.matrix_power(Q, 20)

PROPERTIES

1. Matrix-Matrix Multiplication does not commute.
   $AB \neq BA$, usually

2. is associative.
   $C(B(A\vec{x})) = (CB)(A\vec{x}) \neq ABC$