EECS 126 3.4 LECTURE 13

CLASSIFICATION OF STATES

If there is a path from i to j, then j is accessible from i $(i \to j)$
If we also have $j \to i$, then i and j communicate $(i \leftrightarrow j)$
Conventionally, we say $i \leftrightarrow i$ $\forall i \in S$.

CLAIM $\leftrightarrow$ is an equivalence relation on S, meaning we can partition S into classes of communicating states

$i \leftrightarrow i$
$i \leftrightarrow j \iff j \leftrightarrow i$
if $i \leftrightarrow k$ and $k \leftrightarrow j$, then $i \leftrightarrow j$

A Markov chain is irreducible if it only has one class.

CLASS PROPERTIES

A state i is recurrent if, given $X_0 = i$, the process revisits i wp. 1
Will visit i infinitely many times if you start there, wp. 1
A state i is transient if it is not recurrent.
Will visit i finitely many times if you start there, wp. 1

Let RV $T_i = \min\{n \geq 1 : X_n = i\}$, the first time we enter state i.
If $i \in S$ is recurrent: if $E[T_i \mid X_0 = i] < \infty$, it is positive recurrent
if $E[T_i \mid X_0 = i] = \infty$, it is null recurrent

For $i \in S$, let $\text{period}(i) = GCD\{n \geq 1 : P_{ii}^n > 0\}$
An irreducible MC is periodic if $i \in S$ has period 1.

A probability distribution $\pi = (\pi_i)_{i \in S}$ is said to be a stationary distribution if $\pi = \pi P$ $\left(\pi_j = \sum_i \pi_i P_{ij} \quad \forall j \in S\right)$

If $X_0 \sim \pi$, then $X_n \sim \pi$ $\forall n \geq 0$. $P_r\{X_1 = j\} = \sum P_r\{X_1 = j \mid X_0 = i\} P_r\{X_0 = i\}$

BIG THEOREM

Let $(X_n)_{n \geq 0}$ be an irreducible MC. Then, one of the following is true:

- all states are transient/all states are recurrent.
  no stationary distribution exists and $\lim_{n\to\infty} P_{ij}^n = 0$ $\quad \forall i,j \in S$
- all states are positive recurrent. a stationary distribution π exists, is unique, and satisfies $\lim_{n\to\infty} \frac{1}{n} \sum_{k \in [n]} P_{ij}^k = E[T_i \mid X_0 = i]^{-1}$ $\quad \forall j \in S$
  moreover, if MC is aperiodic, $\lim_{n\to\infty} P_{ij}^n = 0$ $\quad \forall i,j \in S$

Every irreducible finite-state MC is positive recurrent.