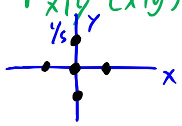
CS 70 ## 4.14 Lecture 23

Joint PMFs

$P_{X,Y}(x,y) = P_r((X=x)\cap(Y=y)) \longrightarrow$ joint

$P_{X|Y}(x|y) = P_{X,Y}(x,y) / P_Y(y) \longrightarrow$ marginal

EX



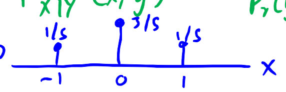
$P_X(x) = \sum_y P_{X,Y}(x,y)$

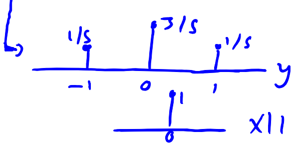
$P_Y(y) = \sum_x P_{X,Y}(x,y)$

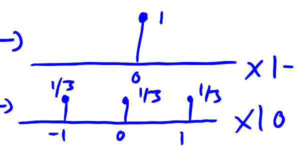
$P_{X|Y}(x,y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$

independent? $P_{X,Y}(x,y) \overset{?}{=} P_X(x)\,P_Y(y)$

for $(x,y)=(1,1)$, $\quad 0 \neq \frac{1}{5}\cdot\frac{1}{5}$







$P_{X|Y}(x|y) = \frac{P_{Y|X}(y|x)\,P_X(x)}{P_Y(y)}$ (Bayes!)

Variance for Sum of RVs

$X = RV \qquad \sigma_X^2 = E(X^2) - E^2(X)$

$Z = X+Y \qquad E(Z) = E(X) + E(Y)$

$\sigma_Z^2 = E(Z^2) - E^2(Z)$

$= E[(X+Y-E(X)-E(Y))^2]$

$\sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY} \longleftarrow = E[((X-E(X))+(Y-E(Y)))^2]$

Covariance

$\sigma_{XY} = \mathrm{cov}(X,Y) = E[(X-E(X))(Y-E(Y))] = E(XY) - E(X)E(Y)$

CLAIM If X,Y indep, then X,Y uncorrelated $(\sigma_{XY}=0)$.

PROOF $P_{X,Y}(x,y) = P_X(x)\,P_Y(y)$

$E[XY] = \sum_x \sum_y g(X,Y)\,P_{X,Y}(x,y)$

$= \sum_x \sum_y xy\,P_X(x)\,P_Y(y)$

$\sigma_{XY} = E(XY) - E(X)E(Y) = 0. \blacksquare \longleftarrow = \sum_x x P_X(x) \sum_y y P_Y(y) = E(X)\,E(Y)$

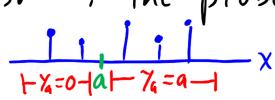
The reverse isn't necessarily true.

Markov Inequality

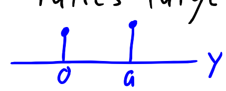
X nonnegative RV $\qquad P(X \geq a) \leq \frac{E(X)}{a} \quad \forall a > 0$

If $E(X)$ is small, the probability that X takes large values is small.

PROOF $Y_a = \begin{cases} 0 & x < a \\ a & x \geq a \end{cases}$



$Y_a \leq X \longrightarrow$



$E(Y_a) = aP(X \geq a)$

$\leq E(X) \quad \blacksquare$

Chebyshev's Inequality

X RV $\qquad \mu = E(X) \qquad \sigma^2 = \sigma_X^2 \qquad P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2} \quad \forall c > 0$

Probability that X deviates from mean by c is bounded by $\frac{\sigma^2}{c^2}$. Prove w/Markov!

PROOF $Y = (X-\mu)^2, \quad a = c^2 \qquad Y \geq 0, \quad P(Y \geq a) = P(Y \geq c^2) \leq \frac{E(Y)}{a} = \frac{E((X-\mu))^2}{c^2}$

$P((X-\mu)^2 \geq c^2) \leq \frac{\sigma^2}{c^2}$ the left side here is the same as $P(|X-\mu| \geq c) \quad \blacksquare$