CS 70 ## 3.31 LECTURE 19 ##

MUTUAL INDEPENDENCE

Pairwise doesn't imply mutual, but mutual does imply pairwise.

Events A_1 & A_2 are independent is $P(A_1 \cap A_2) = P(A_1)P(A_2)$. For n events, $A_1 \ldots A_n$ are mutually independent if for every subset $S \in \{1, \ldots, n\}$

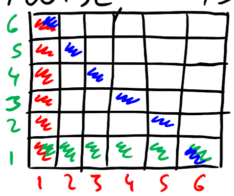
$P(\bigcap_{i \in S} A_i) = \prod_{i \in S} P(A_i)$

EX Events A_1, A_2, A_3; must check: $\binom{3}{2} = 3$: $P(A_1 \cap A_2) = P(A_1)P(A_2)$, $P(A_1 \cap A_3)$, $P(A_2 \cap A_3)$;

$\binom{3}{3} = 1$: $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

For pairwise, ⌐ is true. For mutual, ⅃ must also be true.

EX Dice!



A_1: red die comes up 1 — $P(A_1) = 1/6$ — $P(A_1 \cap A_2) = 1/36$

A_2: green die yields 1 — $P(A_2) = 1/6$ — $P(A_2 \cap A_3) = 1/36$

A_3: sum of dice is 7 — $P(A_3) = 1/6$ — $P(A_1 \cap A_3) = 1/36$

Visibly, $P(A_1 \cap A_2 \cap A_3) = 0 \neq P(A_1)P(A_2)P(A_3) = 1/216$

Here, A_1, A_2, A_3 are pairwise independent, but not mutually independent.

RANDOM VARIABLES

Ω: ω_1, ω_2, ω_3, ... → x_1

$X : \Omega \to \mathbb{R}$, $X(\omega) = x$

The RV X is a function whose domain is Ω & range is $\mathbb{R}$; must be surjective.

EX Roll a 6-sided die & read the top number. $\Omega = \{1, \ldots, 6\}$, R = value

Probability Mass Function (PMF) Distribution: 1 2 3 4 5 6 → r (particular value of RV R)

This is called the Uniform PMF, where each reachable outcome is equally likely.

$P_R(r) = P_r(R=r) = P(R=r)$, probability that random var. R takes on value r.

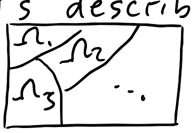
BERNOULLI RV

Takes on value of 0 or 1.

EX Coin toss! p: 1, $1-p$: 0 — K: Bernoulli RV — 0 1 → K ← the heights correspond to P_K, and may not be exactly equal.

PMFs describe probabilities, so their values are non-negative & sum to 1.

EX Ω



$\bigcup_{i=1}^{\infty} \Omega_i = \Omega$, $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$ } $X(\omega) = x_i$ if $\omega \in \Omega_i$, $P_X(x_i) = P_r(\omega \in \Omega_i) = P(\Omega_i)$

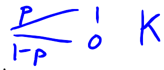
Here, $\sum_{i=1}^{\infty} P_X(x_i) = \sum_{x_i} P_X(x_i) = \sum_i P(\Omega_i) = 1$

Binomial RV

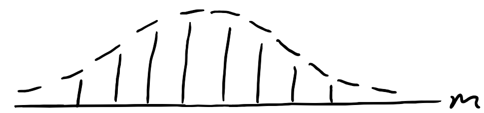
n Bernoulli, and we consider the number of "successes" (either 0 or 1)

$P_M(m) = \binom{n}{m}(1-p)^{n-m}\,p^m \qquad m \in \{0, \dots, n\}$

↳ n bits, m of which are 1



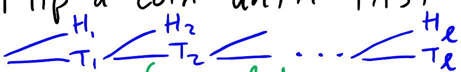
↳ for large n's, →



$M = K_1 + \dots + K_n$

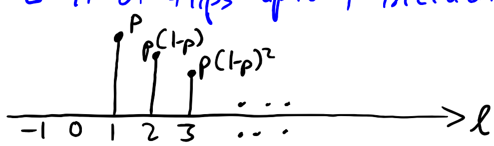
Geometric RV

Flip a coin until first head ("success"), then stop.



L: # of flips up to & including first head

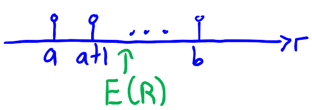
$$P_L(\ell) = \begin{cases} p(1-p)^{\ell-1} & \ell > 0 \\ 0 & \ell \le 0 \end{cases}$$



$$\sum P_L = p\sum (1-p)^{\ell-1} = p\sum (1-p)^m = p\,\frac{1}{1-(1-p)} = p\,\frac{1}{p} = 1$$

The Bernoulli, Binomial, and Geometric RVs are part of a family, as they all stem from the Bernoulli RV.

Expected Value: Uniform



$P_R(r) = \frac{1}{b-a+1}$

$E(R)$ is the expected center of mass

$E(R) = \frac{a+b}{2}$

For a 6-sided die, the expected value would be $\frac{1+6}{2} = 3.5$.