CS 70 ##3.19 LECTURE 18##

CONDITIONAL PROBABILITY

$P(A|B) = P(A \cap B)/P(B) \qquad P(B) \neq 0$

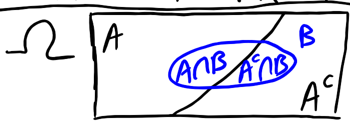
if A & B are independent, $\underbrace{P(A|B)}_{\text{conditional}} = \underbrace{P(A)}_{\text{a priori}}$

$P(A \cap B) = P(A|B)P(B) \longrightarrow P(A \cap B) = P(A)\,P(B)$ if independent

EX $\quad A \cap B = \emptyset,\ P(A) \neq 0,\ P(B) \neq 0 \qquad P(A \cap B) = P(\emptyset) = 0 \neq P(A)P(B)$

so A & B must be dependent, in this case mutually exclusive!

LAW OF TOTAL PROBABILITY (revisited)

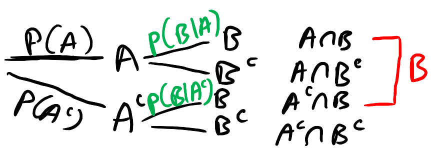


In terms of conditional probability,

$P(B) = P(A \cap B) + P(A^c \cap B) = P(B|A)P(A) + P(B|A^c)\,P(A^c)$

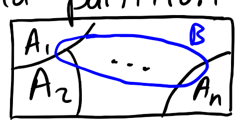
$= P(A)\,\frac{P(B \cap A)}{P(A)} + P(A^c)\,\frac{P(B \cap A^c)}{P(A^c)}$ ✓

Tree diagram!



notice that this diagram is consistent with the law above.

Extend partition to event space $\Lambda = \{A_1, \ldots, A_n\} \quad \bigcup_{i=1}^{n} A_i = \Omega \quad A_i \cap A_j = \emptyset$



$P(B) = P(B \cap A_1) + \ldots + P(B \cap A_n)$

$= \sum_{i=1}^{n} P(A_i)P(B|A_i)$

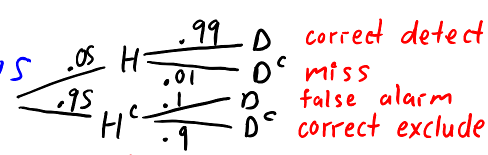
BAYES'S RULE

$P(A|B) = P(B|A)P(A)/P(B)$

EX $\quad$ Medical Diagnostic Test (HIV Testing)

H: a person is actually infected $\longrightarrow P(H) = 0.05$

D: test results positive



Say a test comes out positive. Firstly, what is $P(\text{Error})$? $P(H \cap D^c) + P(H^c \cap D)$

$= 0.05 \cdot 0.01 + 0.95 \cdot 0.1$

≈ 0.1

Next, what is $P(H|D)$? $P(D|H)\,P(H)/P(D)$

$= 0.99 \cdot 0.05/(P(H \cap D) + P(H^c \cap D))$

$= 0.99 \cdot 0.05/(0.05 \cdot 0.99 + 0.95 \cdot 0.1)$

$= .0495/.1445 \approx 0.3425$

So, the probability that you actually have HIV if the test is positive is 34.3%. Pretty low!

Detour: Conditional Independence

Unconditional $\longrightarrow$ $P(A \cap B) = P(A)\,P(B)$

Conditional $\longrightarrow$ A & B are independent conditioned on C if

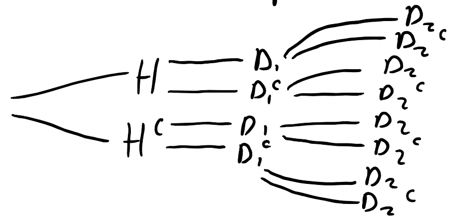
$$P(A \cap B \mid C) = P(A \mid C)\,P(B \mid C)$$

Note that events may be neither, one of, or both unconditionally and conditionally independent.

HIV Test (revisited)

Let's try to improve the outcome. We don't have resources to create a whole new test, so we need to update this one.

We can run the test again!



each test is independent on H, so $P(D_2 \mid H \cap D_1) = P(D_2 \mid H) = .99$

What should we use to declare someone HIV positive? 2 positive tests!

What is $P(H \mid D_1 \cap D_2)$? $P(H \cap D_1 \cap D_2) / P(D_1 \cap D_2) = \dfrac{0.05 \cdot 0.99 \cdot 0.99}{0.05 \cdot 0.99^2 + 0.95 \cdot 0.1^2} = \dfrac{0.049}{0.059}$

So, the probability that you have HIV if you test positive twice is 83.8%. Much better!