CS 70 — ## 3.10 LECTURE 15

$\binom{2n}{n} = (2n)! / (n!n!)$ $C_n = \frac{1}{n+1}\binom{2n}{n}$

Stirling: $n! \approx \sqrt{2\pi}\sqrt{n}\left(\frac{n}{e}\right)^n \longrightarrow \binom{2n}{n} = 2^{2n}/\sqrt{\pi n}$ Then, $C_n \approx O\left(2^{2n}/n^{3/2}\right)$ rough behavior

COUNTABILITY

Zeroeth Rule: $|A| = |B|$ iff $\exists$ bijection $h: A \to B$

↳ one-to-one (injective) & onto (surjective)

$|A| \le |B|$ iff $\exists$ a one-to-one map $f: A \to B$

Cantor-Schröder-Bernstein Theorem: $|A| \le |B| \wedge |B| \le |A| \implies |A| = |B|$

see Hilbert's Hotel for an application of infinity & infinite sets

this is the classic story regarding cardinality

Cantor's Diagonalization Proof: $2^\infty > \infty$ how many reals in interval $[0,1]$?

PROOF assume there exists a bijective map $f: \mathbb{N} \to [0,1]$. arrange the decimal expansions: $f(0) = 0.\,f(0)[0]\,f(0)[1]\,f(0)[2]\ldots$, want to construct

$f(1) = 0.\,f(1)[0]\,f(1)[1]\,f(1)[2]\ldots$

$q \notin f : 0.\,\overline{f(0)[0]}\,\overline{f(1)[1]}\,\overline{f(2)[2]}\ldots$ s.t. $\bar{0}=1, \bar{1}=2, \ldots, \bar{8}=9, \bar{9}=1$

↳ $q = \sum_{i=1}^{\infty} 2^{-i} f(i-1)[i-1] \longrightarrow q$ differs from $f(i)$ in the i^{th} decimal place, so there is no j s.t. $f(j) = q$, so f can't exist. $10^\infty > \infty$. ▨

COMPUTABILITY

Halting Problem. Programs are natural numbers.

Properties of programs are infinite bit strings:

$0.P_1P_2P_3\ldots$ $P_i = 1$ if Program i has property, else 0.

Because $|[0,1]| > |\mathbb{N}|$, there are properties of programs that can't be checked by a program.

Does a program halt or not? Uncomputable.

PROOF assume TestHalt$(P, x) = \{$"yes" if P halts on input x, "no" otherwise$\}$.

then we can write Turing$(P) = \{$loop forever if TestHalt$(P,P) =$ "yes", else halt$\}$

what does Turing(Turing) do? if TestHalt says it'll halt, it doesn't.

if TestHalt says it won't, it'll halt.

so either TestHalt is wrong, or it can't exist.