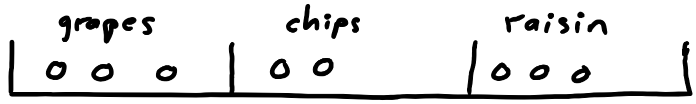
CS 70 ##3.5 LECTURE 14

COMBINATORIAL PROOFS

EX. With your frozen yogurt, you can take n toppings. Each topping can be 1 of 3 types k. How many ways? Add detail!



tray with $n+2$ slots

we only need to know the position of the dividers $\rightarrow \binom{n+2}{2}$ ways to place 2 dividers in $n+2$ slots. this leaves n slots for toppings!

For a general k, $\binom{n+k-1}{k-1}$. This is the heart of combinatorial proofs: changing the story to an analogous one. This can help prove identities!

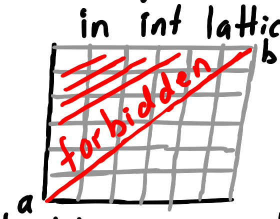
EX. Prove $2^n = \sum_{i=0}^{n} \binom{n}{i} \rightarrow 2^n$ is a binary string of length n. $\binom{n}{i}$ is # of such n-length strings with exactly i 1s.

EX. Prove $\binom{n}{k+1} = \sum_{i=1}^{n-k} \binom{n-i}{k} \rightarrow$ choose $k+1$ things out of n. say $n = \{1, 2, 3, \ldots, n\}$. (because $1=1$)
In this story, what is i? Say it's the smallest # you pick, which must $\leq n-1$. Then, $n-i$ is numbers left to choose from. Instead of ways to choose $k+1$, we choose a smallest and then k.

EX. This example's techniques are out of scope, but is fun & complex.
Catalan Numbers: how many ways can you have a string of length $2n$ that consists only of (,) with balanced parentheses?
Note: 2^{2n} strings of {(,)} of length $2n$. $\binom{2n}{n}$ strings w/ n (& n) exactly.
The more (you add, the more freedom you have for a while. New story:

integer line: (is right,) is left. C_n is # of paths in int lattice starting @ 0 that can't go negative.

New story: ($= \rightarrow$,) $= \uparrow$.



C_n is # of ⌐ paths a to b that don't cross diagonal $\rightarrow$ "good paths"
Observe that good $=$ all $-$ bad $= \binom{2n}{n} -$ bad

Bad paths must enter forbidden zone. How do we count them?
If you try it for $n = 2, 3, 4, \ldots$, it'll look like $\binom{a}{b}$. But we'll use WOP:
All bad paths have a first bad move.

Before	Bad Move	After	Flipped	End
$k \rightarrow$ $k \uparrow$	$1 \uparrow$	$n-k \rightarrow$ $n-k-1 \uparrow$	$n-k-1 \rightarrow$ $n-k \uparrow$	$(n-1, n+1)$

All flipped bad paths end up @ $c=(n-1, n+1)$.
All bad paths $\longleftrightarrow$ flipped bad path, & flipped bad paths are subset of all paths w/ $n-1 \rightarrow$ & $n+1 \uparrow$ that go from a to c. However, by the Intermediate Value Thm., all these paths are bad, so $\binom{2n}{n-1} = \#$ bad paths.
\# of good paths, then, is $\binom{2n}{n} - \binom{2n}{n-1}$.

The number of bad paths would be reached by trying bad paths for n.

How big is $\binom{2n}{n} - \binom{2n}{n-1}$? $\frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n+1)!\,(n-1)!} = \frac{(n+1)(2n)! - n(2n)!}{(n+1)!\ n!} = \frac{(2n)!}{(n+1)\,n!\,n!}$
$= \frac{1}{n+1}\binom{2n}{n}$

Is this $\frac{1}{n+1}$ factor big or small?

STERLING'S APPROXIMATION

$n! \approx \sqrt{2\pi \boxed{n}}\ \boxed{\left(\frac{n}{e}\right)^n}$ three irrational #s crash the party.

will derive in next homework :)