CS 70 ## 2.4 LECTURE 5

WELL-ORDERING PRINCIPLE

All non-empty subsets of natural numbers have a minimum.

Are there always stable matches? Can we find them? Propose & Reject

EX Round-robin: n teams face every other team once, no ties

Either consistent results or cycles.

CLAIM If cycle, then $\exists$ cycle of length 3, ie, $t_1 > t_2 > t_3 > t_1$

PROOF Assume no cycles of length 3, but that $\exists$ cycles. By WOP, pick a shortest cycle: $t_{i_1} > t_{i_2} > t_{i_3} > \;\; > t_{i_k} > t_{i_1}$: of length k

By our tournament, t_{i_2} played t_{i_k}. What happened?

CASE $t_{i_2} > t_{i_k}$: $t_{i_1} > t_{i_2} > t_{i_k} > t_{i_2}$ length 3, contradiction

CASE $t_{i_2} < t_{i_k}$: $t_{i_2} > t_{i_3} > \ldots > t_{i_k} > t_{i_2}$ length k-1, contradiction of WOP

Are stable matches unique? No We only use P&R, but there's more options!

OPTIMALITY

Who prefers which matching? Depends!

EX

1	B	A	--		A	1	2	--	
2	A	B	--		B	2	1	--	
3	B	A	D	C	C	1	2	3	4
4	A	B	C	D	D	2	1	4	3

two tiers

two matches per tier

MATCH △ 1A 2B ⋮ 3C 4D ; □ 1A 2B ⋮ 3D 4C

○ 1B 2A ⋮ 3C 4D ; ☆ 1B 2A ⋮ 3D 4C

PREF 1 ○☆, △□   2 ○☆, △□   3 □☆, △○   4 □☆, △○

A △□, ○☆   B △□, ○☆   C △○, □☆   D △○, □☆

All numbers like ☆, all letters like △. Is there a pattern?

DEF i is a feasible partner for x if they're ever stably matched

DEF o is the optimal partner for x if x prefers o over all other i

DEF M is an optimal matching for employers if it's optimal $\forall$ employers

Let's use these definitions.

CLAIM P&R gives an employer optimal matching

PROOF Assume otherwise. At the end, some employer doesn't have optimal employee.

DEF weird days are when an optimal employee rejects an offer

By WOP, there is the first weird day. J rejected by C^*. But C^* is a feasible employee of J. in favor of J^*.

so $\exists$ match m which is stable & has (J, C^*) in it (& $J^*, \tilde{C}$)

Here, consider (J^*, C^*). Is this rogue in m? Yes

C^* prefers J^* ✓. Does J^* prefer C^* to $\tilde{C}$? $\tilde{C}$ is feasible for J^*. This is the first weird day, so J^*'s optimal employee did not reject on this day or before. Thus, J^* prefers C^* to all others. So is m stable? No, so there is no weird day, so no J has its optimal employee reject them, so P&R gives an employer optimal matching.

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