CS 70 ## 1.28 LECTURE 3

THM. $\exists\, x, y$ irrational such that x^y is rational

PROOF. $\sqrt{2}^{\sqrt{2}}$ this is either rat'l or irrat'l

rational: done

irrational: know $\sqrt{2}^{\sqrt{2}}$ is irrat'l $\rightarrow \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{2} = 2$ rat'l! ▨

this is a nonconstructive existence proof.

INDUCTION

THM. $n^3 - n$ is divisible by 3.

PROOF. Try a few. $0 \to 0$, $1 \to 0$, $2 \to 6$, $3 \to 24$, $4 \to 60$. Looks true.

Why? $n^3 - n = n(n^2-1) = n(n+1)(n-1) = (n-1)n(n+1)$

Three consecutive numbers? One has to be a multiple of 3!

BASE 0

HYPO. Assume true for some k; $k^3 - k = 3q$ for some q

CONSIDER $k+1 \longrightarrow (k+1)^3 - (k+1) = (k^3 - k) + 3k^2 + 3k = 3q + 3k^2 + 3k$
$= 3(q + k^2 + k)$ ▨

STRONG INDUCTION

Instead of assuming true for k, assume true for everything through k

THM All nat. numbers ≥ 2 can be written as a product of 1+ primes

BASE. 2

HYPO All $2 \leq l \leq k$ can be written as a product of 1+ primes

CONSIDER $k+1 \longrightarrow$ cases: if $k+1$ is prime, ▨, else $\exists$ ab $a \neq 1$ $b \neq 1$ such that $ab = k+1$. Notice $2 \leq a \leq k$ & $2 \leq b \leq k$. Invoke

HYPO: $a = \prod_{i=1}^{n_a} P_a(i)$ $\quad b = \prod_{j=1}^{n_b} P_b(j) \Longrightarrow k+1 = \prod_{i=1}^{n_a} \prod_{j=1}^{n_b} P_a(i) P_b(j)$

NOTICE. Why is $a < k+1$? $b < k+1$? a & b are counting numbers $\neq 1$
$a > 1$, $b > 1$. $k+1 = ab > a, > b$, so $a, b < k+1$ ▨

THM Sum of first n odd numbers is a perfect square

HYPO $\forall n \, \exists q \; \sum_{i=1}^{n} (2i-1) = q^2 \longrightarrow$ assume $\sum_{i=1}^{k} (2i-1) = q^2$

CONSIDER $k+1 \to \sum_{i=1}^{k+1} (2i-1) = (2(k+1)-1) + \sum_{i=1}^{k} (2i-1) = 2k+1 + q^2$

WISH $q^2 = k^2 \longrightarrow 2k+1+q^2 + \; 2k+1+k^2 = (k+1)^2$ ∎